

A Self Consistent Study  
of the Earth's Radiation Belts

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Abstract

In most work on the dynamical equilibrium of the Van Allen Belts it is assumed that the plasma is tenuous, i.e. one neglects second order effects such as the magnetic field generated by the particles in the belt compared to the geomagnetic field. It would seem from the results of these calculations that this is not a good approximation and that the plasma energy/vol. is comparable to the geomagnetic energy/vol. In the present work the radiation belts are treated self consistently. In particular the question of how much plasma could be trapped in the belts if an unlimited source were present is considered. Judging from a preliminary calculation it would seem that the Van Allen Belts are saturated.

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## INTRODUCTION

As a first approximation we may consider the Van Allen belts to be a steady state plasma confined in a bounded volume by the earth magnetic dipole field. The temperature, density and magnetic field strength are such that the plasma may be described adequately by the Vlasov equation. If we further observe that the Larmor radius of the particles is much smaller than the distances within which the magnetic field, particle density etc., change by a significant fraction of themselves we may then use the guiding center approximation. If we assume that the plasma is neutral, and thus there are no electric fields we have in the steady state the following equation for dynamical equilibrium

$$(1) \quad \nabla \cdot \vec{P} = \vec{J} \times (\vec{B} + \vec{B}_0)$$

where  $\vec{J}$  is the current due to the motion of plasma constituents,  $\vec{B}$  the magnetic field generated by the plasma,  $\vec{B}_0$  the external magnetic dipole field, and  $\vec{P}$ . The pressure tensor of the plasma. The pressure tensor is defined by

$$(2) \quad \vec{P} = \sum_{\pm} \int m_{\pm} \vec{v} \vec{v} f_{\pm}(\vec{r}, \vec{v}) d^3 v$$

where the sum is taken over the plasma constituents (electrons and ions) and  $f$  is the distribution function for a given constituent. It is assumed in this approximation that  $\vec{P}$  is a diagonal tensor in a local coordinate system with the z-axis parallel to the magnetic field. It is further assumed that  $P_{xx} = P_{yy} = P_{\perp}$ . For convenience we define  $P_{zz} = P_{\parallel}$ .

$P_{\parallel}$

The current  $\vec{J}$  and the field  $\vec{B}$  are related by Maxwell's equations

$$(3) \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

also

$$(4) \quad \nabla \cdot \vec{B} = 0$$

Clearly these equations alone are not sufficient. The equations cannot tell us how many particles are in the Van Allen belts or even how they are distributed. They specify only certain dynamical and electromagnetic consistency requirements.

In a comprehensive discussion of the subject, Apel, Singer and Wentworth\* have studied in detail certain zero order solutions to the above equations. (One can demonstrate that first order particle orbit theory is equivalent to the above system of equations.<sup>†</sup>) By zero order calculation we mean that  $B$  is neglected in comparison to  $B_0$  in eq. (1). However to obtain a solution representing the observed particle densities it would appear that this assumption may not be a good one.

The zero order calculation indicates that the plasma field may well be comparable to the geomagnetic field. For this reason we address ourselves to the question: how much plasma, with given energy distribution, can be contained within a given bounded volume by a magnetic dipole field? For example, we might ask: how much plasma can be contained between two

\* J. R. Apel, S. F. Singer, and R. C. Wentworth, "Effects of Trapped Particles on the Geomagnetic Field", pp. 131-89 in "Advances in Geophysics, 9, 1962, Academic Press

<sup>†</sup> Conrad L. Longmire, Elementary Plasma Physics, John Wiley & Sons, New York, 1963, Ch. III.

spheres of radius  $r_1$  and  $r_2$ , by a dipole field?

In order to obtain the relation between the number of particles trapped in the dipole field and the pressure distribution we need an equation of state. To this end we take the trace of the pressure tensor defined in eq. (2).

$$(5) \quad \text{Tr } \overleftrightarrow{P} = \sum_{\pm} \int m_{\pm} (v_x^2 + v_y^2 + v_z^2) f_{\pm}(\vec{r}, \vec{v}) d^3 v \\ = 2 (N_+ E_+ + N_- E_-) = 2 N E$$

where  $E_{\pm}$  is the mean energy per ion (electron) and  $N_{\pm}$  the number density of the ions (electrons).  $E$  is the average energy per particle and is defined by

$$E = \frac{N_+ E_+ + N_- E_-}{N_+ + N_-}$$

We have also set  $N = N_+ + N_-$ .

In the Van Allen belts  $N_+ = N_-$  and  $E_+ \gg E_-$ . We will therefore write

$$\text{Tr } \overleftrightarrow{P} \approx 2 N_+ E_+$$

and in the future we will omit the + subscript as there will be no further mention of the electrons.

We shall assume also that  $E$  is a constant throughout the belt and is given (experimentally) so that eq. (5) gives the number density as a function of pressure. ( $N$  and  $P$  remain functions of position.)

Our problem then may be stated as follows. Find a solution of eqs. (1), (3) and (4) with  $\overleftrightarrow{P} = 0$  outside the shell  $r_1 < r < r_2$  such that  $\int \text{Tr } \overleftrightarrow{P} d^3 r$  is a maximum. Our objective is to compare this maximum particle number with the observed particle number to see if the Van

Allen belts are in fact saturated. There are other alternatives of course. The particle density could be limited by virtue of the fact that the particle source is limited so that when equilibrium is reached between the source and sink (capture in the atmosphere, etc.) the density is below the maximum discussed above. Or it is quite possible that instabilities arrive before the maximum is attained. In any case it is worthwhile to examine the problem to see what this maximum is.

We hasten to point out that we were not successful in solving the general problem outlined above. We have limited ourselves to showing that an upper bound exists and deriving bounds in terms of certain plasma parameters (eg. magnetic moment, field generated by the plasma at the earth's surface, etc.). We maximize the density for a two dimensional model by choosing spherical solutions with variable parameters.

#### Self Confinement

Let us first consider whether the problem as posed has a solution, that is there may be no upper bound on the amount of plasma that could be confined by a dipole in the shell  $r_1 < r < r_2$ . Let us suppose there is no upper bound. If we imagine more and more plasma being packed into the shell there comes a time when the magnetic field of the dipole becomes negligible compared to the magnetic field generated by the plasma. We might as well neglect the dipole field, in which case, the question becomes whether the plasma can confine itself. There is a well known theorem due to Schmidt\* that a plasma satisfying the Vlasov equation

\*G. Schmidt, "Virial Theorem for Plasmas", the Physics of Fluids, 1960, pp. 481-2.

cannot contain itself. If eq. (1) (omitting  $B_0$ ) is dotted with  $\vec{r}$  and both side integrated over all space we find

$$\int \text{Tr} \overleftrightarrow{P} d^3r + \frac{1}{2\mu_0} \int B^2 d^3r = 0$$

(This is a special case of the Virial theorem derived by Schmidt). We have used eqs. (3) and (4). From eq. (5) we see that  $\text{Tr} \overleftrightarrow{P}$  is positive definite and thus the sum could not be zero.

Although this argument is not strictly rigorous (it may not be legitimate to neglect  $B_0$  compared to  $B$  even though it is much smaller over most of the plasma) it nevertheless indicates that a dipole can contain only a finite quantity of plasma in a shell.

#### Some General Theorems

We will first derive some general theorems relating an upper bound on the quantity of plasma with other plasma parameters.

We consider first a generalization of a result derived by Dessler and Parker<sup>\*</sup> for some special/geometries and later by Sckopke<sup>†</sup> for an arbitrary geometry. In both cases the results were derived from particle orbit theory and are zero order calculations (i.e.  $B \ll B_0$ ). In what follows we will not require  $B \ll B_0$ .

We dot both sides of eq. (1) and integrate over all space

$$(6) \quad \int \vec{r} \cdot (\nabla \cdot \overleftrightarrow{P}) d^3r = \int \vec{r} \cdot \vec{J} \times \vec{B} d^3r + \int \vec{r} \cdot \vec{J} \times \vec{B}_0 d^3r$$

<sup>\*</sup>Dessler, A. J., and Parker, E. N., "Hydromagnetic Theory of Geomagnetic Storms", Journ. of Geophys. Res., 64, 1959, pp. 2239-52.

<sup>†</sup>N. Sckopke, "A General Relation Between the Energy of Trapped Particles and the Disturbance Field Near the Earth". Journ. of Geophys. Res., 71, 1966, pp. 3125-30.

We will simplify each term in eq. (6) in turn. First we note that

$$\vec{F} \cdot (\nabla \cdot \vec{P}) = \nabla \cdot (\vec{P} \cdot \vec{F}) - T_r \vec{P}$$

Integrating both sides, converting the integral of the divergence into a surface integral at infinity (where the pressure vanishes) we have

$$\int \vec{F} \cdot \nabla \cdot \vec{P} d^3r = - \int T_r \vec{P} d^3r$$

From the definition of  $\vec{P}$  (eq. 5) we have

$$(7) \quad \int T_r \vec{P} d^3r = 2 N_T \bar{E} = 2 E_T$$

where  $N_T$  ( $E_T$ ) is the total number (energy) of particles in the belt and  $\bar{E}$  the average energy per particle.  $\bar{E}$  is calculated by averaging over all the particles in the belt whereas  $E$  (in eq. 5) is calculated by averaging over a small volume element. We have not assumed in eq. (7) that  $N_+ = N_-$  or that  $E$  is a constant over the belt.

Similarly one may show, using eqs. (3) and (4) that

$$\int \vec{F} \cdot \vec{J} \times \vec{B} d^3r = \frac{1}{2\mu_0} \int B^2 d^3r$$

The last term in eq. (6) may be simplified by noting that the dipole field ( $B_0$ ) may be represented as follows

$$(8) \quad \vec{B}_0 = \frac{\mu_0}{4\pi} \left( 3m \frac{\cos\theta}{r^3} \hat{r} - \frac{m \hat{z}}{r^3} \right)$$

where  $m$  ( $\vec{m} = m\hat{z}$ ) is the magnetic moment and  $\hat{r}$  a unit vector in the radial (vertical) direction. Since  $\vec{r} \cdot \vec{J} \times \hat{r} = 0$  the only term that survives is the second and we have

$$\int \vec{F} \cdot \vec{J} \times \vec{B}_0 d^3r = -\frac{\mu_0 m}{4\pi} \int \frac{\vec{F} \cdot \vec{J} \times \hat{z}}{r^3} = -\frac{\mu_0 m \hat{z}}{4\pi} \cdot \int \frac{\vec{F} \times \vec{J}}{r^3} d^3r$$

But

$$\frac{\mu_0}{4\pi} \int \frac{\vec{r} \times \vec{J}}{r^3} d^3r$$

is the magnetic field due to the plasma currents evaluated at the origin ( $\vec{B}(0)$ ). We have finally from eq. (6).

$$(9) \quad -2 N_T \bar{E} = \frac{1}{2\mu_0} \int B^2 d^3r - \vec{m} \cdot \vec{B}(0)$$

Since  $B^2$  is positive definite we have

$$(10) \quad N_T < \frac{\vec{m} \cdot \vec{B}(0)}{2 \bar{E}}$$

Another interesting consequence of eq. (9) is that  $\vec{m} \cdot \vec{B}(0) > 0$ , that is the plasma field will support the earth's dipole field at the poles and oppose it in the equatorial regions.

We may obtain another upper bound on  $N_T$  by considering the inequality

$$2 \bar{E} N_T < \int \vec{r} \cdot \vec{J} \times \vec{B}_0 d^3r = \int \vec{B}_0 \cdot (\vec{r} \times \vec{J}) d^3r$$

If we assume azimuthal component and so

$$2 \bar{E} N_T < \int B_{0,\theta} r J d^3r$$

where  $B_{0,\theta}$  is the  $\theta$  component of the dipole/field. Since  $B_{0,\theta}$  is positive definite

$$2 \bar{E} N_T < (rJ)_{\max} \int B_{0,\theta} d^3r$$

where  $(rJ)_{\max}$  is the maximum of the product, and the integral of  $B_{0,\theta}$  is taken over the plasma volume ( $V_p$ ) only. Noting that  $B_0 \geq B_{0,\theta}$  and writing  $\bar{B}_0$  for the average of  $B_0$  over the plasma volume we have

$$(11) \quad N_T < (m_p)_{\max} \bar{B}_0 / \bar{E}$$

where  $(m_p)_{\max}$  is given by

$$(m_p)_{\max} = \frac{1}{2} (rJ)_{\max} V_p$$

This is in a manner of speaking the converse of eq. (10) where we have a product of the dipole magnetic moment and the plasma field whereas in eq. (11) we have a product of the plasma magnetic moment with the dipole magnetic field. It is clear that eq. (11) is neither as useful nor as accurate an upper bound on the total number of particles.

### A Simplified Model

In the preceding section we derived some theorems for the upper bound on the number of particles in the Van Allen belts. These upper bounds were given in terms of observables associated with the belt, e.g.,  $B(0)$ . However we have not obtained an upper bound for  $B(0)$ . If we can measure  $B(0)$  (and  $E$ ) we know that there are less than  $\frac{1}{m} \cdot \frac{B(0)}{2E}$  particles in the belt. This does not tell us how many particles could be put in the belt. To answer this question we have had to consider a simplified model. Let us assume that all particles move in the equatorial plane. All the forces acting on the particle in this idealization will be in the plane and thus a self-consistent (though unstable) solution exists. This reduces the complexity of the problem considerably and although we were still unable to give a completely satisfactory answer for the upper bound on the particle number, we were nevertheless successful in obtaining a lower limit for the upper bound, i.e., we will show that the maximum must be greater than a given number.

If we assume cylindrical symmetry about the dipole axis, the equation for equilibrium (eq. 1) becomes

$$(12) \quad \frac{dP_{rr}(r)}{dr} = J(r) [B_z(r) + B_0(r)]$$

where  $P_{rr}$  ( $P_{rr} = P_{\theta\theta}$ ,  $P_{zz} = 0$ ) is the indicated component of the pressure tensor,  $J(r)$  is the component of the plasma current,  $B_z$  is the z-component of the magnetic field generated by the plasma and  $B_0(r)$  is the earth's magnetic field in the equatorial plane. The currents  $J$  and the field  $B$  are related by Maxwell's equations

(13)

We may obtain the number density from eq. (5)

$$(14) \quad P_{rr} = N\bar{E}$$

Our problem of maximizing the amount of plasma that may be confined in a bounded domain ( $r_1 < r < r_2$ ) by a dipole field may be formulated as follows. Find a  $P_{rr}(r)$  satisfying eq. (12) (with  $J(r)$  and  $B_z(r)$  solutions of eqs. (13)) which maximizes the integral

$$(15) \quad \int_{r_1}^{r_2} r P_{rr} dr$$

subject to the constraints  $P_{rr} = 0$  for  $r \leq r_1$  and  $r \geq r_2$  and  $P_{rr}(r) \geq 0$  for  $r_1 \leq r \leq r_2$ . This maximum number is then

$$(N_T)_{\max.} = \frac{2\pi}{E} \max. \int_{r_1}^{r_2} r P_{rr} dr$$

We have not determined the general solution to this problem. We have obtained limited success by picking a pressure  $P_{rr}(r)$  with variable parameters satisfying the constraints and the equation of equilibrium and adjusting the parameters to maximize the integral in eq. (15). This will give  $(N)_{\max.}$  for a restricted class of pressure profiles. Before proceeding with this plan we would like to reformulate the problem to facilitate the calculation. First we may solve eqs. (13) to give  $B_z(r)$

$$(16) \quad B_z(r) = \int_0^\infty J(r') B_R(r, r') dr'$$

where  $B_R(r, r')$  is the magnetic field at a distance  $r$  from the center of a unit current in a ring of radius  $r'$ . The observation point lies in the plane of the ring. Eq. (16) is just the statement that the field due to a planar current distribution  $J(r')$  is that of a sum of ring currents

$J(r')dr'$ . The function  $B_R(r, r')$  is given by Smythe\*

$$B_R(r, r') = \frac{\mu_0}{2\pi} \frac{1}{r+r'} \left[ K\left(\frac{2\sqrt{rr'}}{r+r'}\right) + \frac{r+r'}{r'-r} E\left(\frac{2\sqrt{rr'}}{r+r'}\right) \right]$$

where  $K$  and  $E$  are complete elliptic integrals of the first and second kind. There are several directions in which one might proceed. We chose to eliminate  $B_z(r)$  between eqs. (12) and (16) to obtain the integral equation

$$(17) \quad \frac{dP}{dr} = J(r) \left( \int_0^\infty J(r') B_R(r, r') dr' + B_0(r) \right)$$

If the pressure profile were given the solution of this nonlinear integral equation would determine  $J(r)$ . Needless to say such an equation cannot be solved. It is however a simple matter to solve eq. (17) for  $P(r)$  if the current is given (provided of course that the integral can be performed). As an example that allows an analytic solution (for most current distributions the integral must be performed numerically) we consider a current given by

$$J(r) = J_0 (a_2 - r)(r - a_1) \quad a_1 \leq r \leq a_2$$

$$J(r) = 0 \quad \text{elsewhere}$$

where  $a_1 < a_2$ . We define  $\Delta a = a_2 - a_1$ , and assume  $\Delta a$  much less than  $a_1$ .

We must choose a current distribution that is continuous or the function  $\int_0^\infty J(r') B_R(r, r') dr'$  is singular. Such a current will give a pressure which is constant inside a ring of radius  $r = a_1$  and zero outside the ring  $r = a_2$ . (We have also considered the case of a hollow core, as we find

\* W. Smythe, "Static and Dynamic Electricity", pp. 270-1, McGraw-Hill Book Co., 1950

in the actual Van Allen belts, but the numbers are very much the same.)

Integrating eq. (17) we have

$$(18) \quad P(r) \approx \frac{\mu_0 M J_0 (\Delta a)^3}{24 \pi a_1^3} - \frac{(\Delta a)^6 \mu_0 J_0^2}{144 \pi a_1} \left( \ln \frac{8a_1}{\Delta a} + \frac{7}{4} \right) \quad 0 < r < a_1$$

$$P(r) = 0$$

$$r > a_1$$

To obtain this result we have used an asymptotic expansion for  $B_R(r, r')$  valid when  $r \approx r'$ . Eq. (18) requires then that  $\Delta a$  be small.

Maximizing the pressure with respect to the parameter  $J_0$

$$(19) \quad P_{\max} = \mu_0 M^2 / 16 \pi a^5 \left( \ln \frac{8a_1}{\Delta a} + \frac{7}{4} \right)$$

Unfortunately the pressure in eq. (18) increases indefinitely as a function of  $\Delta a$ . However, the pressure maximized with respect to  $J_0$  (eq. 19) is not a sensitive function of  $\Delta a$ . We therefore choose  $\Delta a$  in eq. (19) as large as possible consistent with the approximation made in eq. (18). We have chosen  $\Delta a = .1a$ . We have then, from the equation of state (eq. (14)), the maximum number density (number per unit area).

$$N_{\max} = \mu_0 M^2 / 16 \pi a^5 \bar{E} \left( \ln 80 + \frac{7}{4} \right)$$

and

$$N_{\text{Total}} = \pi a^2 N_{\max} = \mu_0 M^2 / 16 a^3 \bar{E} \left( \ln 80 + \frac{7}{4} \right)$$

If we set  $a = 2.8 \times 10^7$  m. (approximately the distance to the outside of the Van Allen belts) and  $\bar{E} = 85$  MeV (the mean energy of the protons in the belt) we find for  $N_{\text{total}}$

$$(20) \quad N_{\text{Total}} = .28 \times 10^{27} \text{ particles}$$

We may compare this result with experiment in the following way. In the outer belt there are approximately  $3 \times 10^4$  particles/ $m^3$ . If the particles were distributed uniformly inside a sphere of radius  $2.8 \times 10^7 m$  there would be  $2.7 \times 10^{27}$  particles. Thus we see that the maximum number of particles in this model is less than the observed number. Certainly the model is not a particularly good one but it at least suggests to us that the Van Allen belts are saturated.

In the above calculation a very restricted current density profile was chosen. To be certain that we have a good estimate for the upper bound (in the planar model) we liberalized the possible distributions by considering a current with five parameters

$$(21) \quad \begin{aligned} J(r) &= \sum_{n=1}^5 c_n (r-a)^n & a \leq r \leq b \\ J(r) &= 0 & \text{elsewhere} \end{aligned}$$

We have chosen a function which automatically gives  $J = 0$  at the inner edge of the belt ( $r = a$ ). We impose a constraint on the  $c_n$ 's so that  $J$  is also zero at the outer edge (say  $r = b$ ). Thus

$$\sum_{n=1}^5 c_n (b-a)^n = 0$$

Through a somewhat laborious numerical calculation, taking care near the singularity of  $B_R(r, r')$  and restricting ourselves to positive pressure it is possible to maximize  $N_{\text{total}}$  for current distributions of the form given in eq. (21). We find for  $N_{\text{total}}$

$$(22) \quad N_{\text{Total}} = 1.0 \times 10^{27} \text{ particles}$$

Of course this result is larger than that given by eq. (20) since we have chosen a current distribution of greater flexibility.

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